1. Continuous Time [Optimal Control]

Use Kuhn-Tucker w/ infinitesimally short time period. This application is called <u>Maximum Principle of Pontryagin</u>

$$\max_{c_t} \int_0^\infty e^{-\rho t}\, u(\underbrace{c_t}_{\text{Control}}, \underbrace{k_t}_{\text{endog state}}, \underbrace{x_t}_{\text{Exog state}})\, dt$$

s/c $\dot{k}_t = g(c_t, k_t, x_t)$ ⟵ accumulation (transition) equation

$k_0 > 0$ given ⟵ initial Cond'

$\lim_{t\to\infty}(k_t\, e^{-\rho t}) \geq 0$ ⟵ Terminal Cond' [value of k stock must be Asymptotically zero, Nothing valuable is left over]

Cookbook procedure:

① Construct $H = e^{-\rho t} u(c_t, k_t, x_t) + \underbrace{\mu_t}_{\mu_t:\ \text{value of } k \text{ stock at time } t \text{ in units of time-zero utils}} \cdot g(k_t, c_t, x_t)$

② rewrite $H = e^{-\rho t}\left[u(c_t, k_t, x_t) + q_t\, g(k_t, c_t, x_t)\right]$ where $q_t \equiv e^{\rho t}\mu_t$ = current-value shadow price

③ rewrite $\hat{H} = H e^{\rho t} = \left[u(c_t, k_t, x_t) + q_t \cdot g(k_t, c_t, x_t)\right]$

(4). $\hat{H}_c = 0$

$\boxed{5}$ $\hat{H}_k = \rho q - \dot{q}$ [Asset-pricing]

$\boxed{6}$ (TV) $q(T) \cdot e^{-\rho T} \cdot k(T) = 0$

$\lim_{t \to \infty} [\mu(t) \cdot k_t] = 0$ [with discounting]

$\lim_{t \to \infty} [H_t] = 0$ [without discounting] [Michel's Condition]

use $\boxed{4}$ and $\boxed{5}$ to get a system of two differential equations in c and k.

Example

$$\max \int_0^\infty e^{-\rho t} \frac{c_t^{1-\theta}}{1-\theta}\, dt$$

neoclassical growth with fixed Labor Supply

s/c $\dot{K}_t = A K_t^{1-\alpha} N^\alpha - c_t - \delta K_t$

$K(0) = K_0 > 0$ given

Typos : forgot Kdot in the constraint

[1] $H = \frac{c_t^{1-\theta}}{1-\theta} + \lambda_t \left[A K_t^{1-\alpha} N^\alpha - c_t - \delta K_t \right]$

[2] $H_c = c_t^{-\theta} - \lambda_t = 0$

[3] $H_K = \lambda_t \left[(1-\alpha) A K_t^{-\alpha} N^\alpha - \delta \right] - \rho \lambda_t = -\dot{\lambda}_t$

[4] $H_\lambda = A K_t^{1-\alpha} N^\alpha - c_t - \delta K_t = \dot{K}_t$

[5] $\lim_{t \to \infty} \lambda_t k_t = 0$

Combine $\longrightarrow$ find that it is a system of 2 differential equations in $\underline{\dot{c} \text{ and } \dot{K}}$