Real Business Cycles
Class Notes for ECO 7205

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1 Real Business Cycle model

Lucas (1977) defined 'business cycles' as the fluctuations of output about trend and its co-movements with other aggregate variables.[1] As mentioned earlier, Frisch (1933) provided an early theoretical theory of the business cycle. Keynesian oriented multiple system of equations[2] dominated the 1960's and the 1970's. These were followed in the early nineteen eighties by a new class of models based on Walrasian analysis.

The Real Business Cycle model[3] (RBC) is based on the neoclassical growth model and stochastic dynamic programming, (Kydland and Prescott (1982)). The idea of the basic RBC model is as follows. Adding a stochastic element to a standard aggregative growth model allows for changes in productivity. After calibrating the model's microeconomic parameters, stochastic simulations of this model produce time series for output, employment, consumption and investment. The characteristics of the moments of the simulated data are then matched to their counterpart in the business cycle data. In the RBC framework, economic agents are subjected to various types of shocks and take optimal decisions in a dynamic environment.

2 General RBC (Uhlig)

This section follows closely the RBC model derived in Uhlig (1997). For the basic stochastic neoclassical growth model, the environment is as follows.

1) Preferences: A representative agent maximizes his expected utility

$$U = E \sum_{t=0}^{\infty} \left[\beta^t \frac{C_t^{1-\eta} - 1}{1-\eta} \right] \tag{1}$$

where C_t is consumption, $0 < \beta < 1$ is the discount factor and $\eta > 0$ is the coefficient of relative risk aversion. β is equal to $1/(1+r)$ where r is the pure rate of time preference.

[1] This definition was referred to as 'the business cycle phenomena' in Prescott (1986).

[2] Cooley and Prescott (1995, p. 3), referred to these models as "... fully specified artificial economies ...". Those type of models were engineered to study static output determination.

[3] Or as I prefer to use [In the language of Lucas (1980)] "... fully articulated, artificial economic system ..." (p. 696).

2) The technology: Firms have a Cobb-Douglas production function

$$Y_t = Z_t K_{t-1}^{\rho} N_t^{1-\rho} \tag{2}$$

where K_t and N_t are capital and labour, respectively. $0 < \rho < 1$ is capital's share in production and Z_t is the exogenous total factor productivity.

3) The laws of motion that describe how capital and technology evolve through time: For capital, the dynamic equation is

$$K_t = (1-\delta)K_{t-1} + I_{t-1} \tag{3}$$

For the technology shock, the equation is

$$\log Z_t = (1-\psi)\log \overline{Z} + \psi \log Z_{t-1} + \varepsilon_t \tag{4}$$

where $\varepsilon_t \sim iid \quad N\left(0;\sigma^2\right)$ and $0 < \psi < 1$.

4) Endowment: In each period, the representative household is endowed with one unit of time so that $N_t = 1$ for all t. Also, K_0 is set equal to zero.

5) The information set: The representative household chooses C_t, N_t and K_t given the above information up to time t.

Since there are neither externalities nor distortionary taxes in this economy, the social planner's solution will be the same as the competitive equilibrium.

2.1 The social planner problem

The problem presented for the social planner is to maximize expected utility (equation (1)) subject to the feasibility constraints. That is,

$$\max_{(C_t,K_t)_{t=0}^{\infty}} E\sum_{t=0}^{\infty}\left[\beta^t \frac{C_t^{1-\eta}-1}{1-\eta}\right] \tag{5}$$

subject to

$$C_t + K_t = Z_t K_{t-1}^{\rho} N_t^{1-\rho} + (1-\delta)K_{t-1} \tag{6}$$

$$\log Z_t = (1-\psi)\log \overline{Z} + \psi \log Z_{t-1} + \varepsilon_t \tag{7}$$

$$\varepsilon_t \sim iid \quad N\left(0;\sigma^2\right) \tag{8}$$

To solve this problem, one can apply the techniques of dynamic programming (section 2.3) or use the Lagrangian method.

2.2 Lagrangian method

The Lagrangian for the above problem is,

$$L = \max_{\{C_t, K_t\}_{t=0}^{\infty}} E\left[\sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\eta} - 1}{1-\eta} - \lambda_t (C_t + K_t - Z_t K_{t-1}^{\rho} N_t^{1-\rho} - (1-\delta) K_{t-1})\right)\right] \tag{9}$$

Its first order conditions (FOC) (called also the Euler equations) are,

$$\frac{\partial L}{\partial \lambda_t} \quad : \quad C_t + K_t - Z_t K_{t-1}^{\rho} N_t^{1-\rho} - (1-\delta) K_{t-1} = 0 \tag{10}$$

$$\frac{\partial L}{\partial C_t} \quad : \quad C_t^{-\eta} - \lambda_t = 0 \tag{11}$$

$$\frac{\partial L}{\partial K_t} \quad : \quad -\lambda_t + \beta E_t \left[\lambda_{t+1} (\rho Z_{t+1} K_t^{\rho-1} + (1-\delta))\right] \tag{12}$$

The transversality condition - to rule out explosive solutions - is,

$$\lim_{T \to \infty} E_0 \left[\beta^T C_T^{-\eta} K_T\right] = 0 \tag{13}$$

It is obtained by summing the planner's problem for T periods rather than for ∞ (i.e., obtained from limiting the Kuhn-Tucker condition).

2.2.1 The steady state

To solve for the steady state, rearrange the FOC such that,

$$C_t = Z_t K_{t-1}^{\rho} N_t^{1-\rho} + (1-\delta) K_{t-1} - K_t \tag{14}$$

$$R_t = \rho Z_t K_{t-1}^{\rho-1} + (1-\delta) \tag{15}$$

$$1 = E_t \left[\beta \left(\frac{C_t}{C_{t+1}}\right)^{\eta} R_{t+1}\right] \tag{16}$$

$$\log Z_t = (1-\psi) \log \overline{Z} + \psi \log Z_{t-1} + \varepsilon_t \tag{17}$$

Equation (16) is the Lucas asset pricing equation. Now, drop the subscript t, and replace the variables with their steady state values. For any variable M_t which denotes the level, let $\overline{M}$ denote the steady state value, and $m_t = \log M_t - \log \overline{M}$ denote the deviation from its steady state value. If $m_t = 0.05$, then M_t is 5 percent above its steady state value. Re-writing the FOC in steady state yield,

$$\overline{C} = \overline{Z}\,\overline{K}^{\rho} - (1-\delta)\overline{K} - \overline{K} \tag{18a}$$

$$\overline{R} = \rho \overline{Z}\,\overline{K}^{\rho-1} + (1-\delta) \tag{18b}$$

$$1 = \beta \overline{R} \tag{18c}$$

Solving each steady state variable as a function of the parameters of the model and $\overline{Z}$,

$$\begin{aligned}
\overline{R} &= 1/\beta && (19)\\
\overline{K} &= \left(\frac{\rho\overline{Z}}{\overline{R}-1+\delta}\right)^{1/(1-\rho)} && (20)\\
\overline{Y} &= \overline{Z}\,\overline{K}^{\rho} && (21)\\
\overline{C} &= \overline{Y}-\delta\overline{K} && (22)
\end{aligned}$$

2.2.2 Log-linearization

The following log-linearizes the FOC around the steady states:

1) For equation (14),

$$C_t = Z_t\, K_{t-1}^{\rho} + (1-\delta)\, K_{t-1} - K_t \tag{23}$$

replace each variable by its steady state, using $M_t = \overline{M}e^{m_t}$,

$$\overline{C}e^{c_t} = \overline{Z}\,\overline{K}^{\rho}e^{z_t+\rho k_{t-1}} + (1-\delta)\,\overline{K}e^{k_{t-1}} - \overline{K}e^{k_t} \tag{24}$$

and $\overline{M}e^{m_t} \approx \overline{M}(1+m_t)$,

$$\overline{C}+\overline{C}c_t \approx \overline{Z}\,\overline{K}^{\rho}+(1-\delta)\overline{K}-\overline{K}+\overline{Z}\,\overline{K}^{\rho}(z_t+\rho k_{t-1})+(1-\delta)\overline{K}k_{t-1}-\overline{K}k_t \tag{25}$$

Since $\overline{Y}=\overline{Z}\,\overline{K}^{\rho}$ and $\overline{C}=\overline{Y}-\delta\overline{K}$,

$$\overline{C}c_t \approx \overline{Z}\,\overline{K}^{\rho}(z_t+\rho k_{t-1}) + (1-\delta)\overline{K}k_{t-1} - \overline{K}k_t \tag{26}$$

Dividing by $\overline{C}$,

$$c_t \approx \frac{\overline{Y}}{\overline{C}}z_t + \frac{\overline{K}}{\overline{C}}\overline{R}k_{t-1} - \frac{\overline{K}}{\overline{C}}k_t \tag{27}$$

2) For equation (15),

$$\begin{aligned}
R_t &= \rho Z_t K_{t-1}^{\rho-1} + 1 - \delta && (28)\\
\overline{R}\,e^{r_t} &= \rho\overline{Z}\,\overline{K}^{\rho-1}\; e^{z_t+(\rho-1)k_{t-1}} + 1 - \delta && (29)\\
\overline{R}+\overline{R}r_t &\approx \rho\overline{Z}\,\overline{K}^{\rho-1} + 1 - \delta + \rho\overline{Z}\,\overline{K}^{\rho-1}(z_t+(\rho-1)k_{t-1}) && (30)
\end{aligned}$$

using $1/\beta = \overline{R} = \rho\overline{Z}\,\overline{K}^{\rho-1} + 1 - \delta$ (Equation (18b)),

$$\overline{R}r_t \approx \rho\overline{Z}\,\overline{K}^{\rho-1}(z_t+(\rho-1)k_{t-1}) \tag{31}$$

so that

$$r_t \approx (1-\beta(1-\delta))(z_t-(1-\rho)k_{t-1}) \tag{32}$$

3) For equation (16),

$$1 = E_t\left[\beta\left(\frac{C_t}{C_{t+1}}\right)^{\eta} R_{t+1}\right] \tag{33}$$

$$1 = E_t\left[\beta\left(\frac{\overline{C}e^{c_t-c_{t+1}}}{\overline{C}}\right)^{\eta} \overline{R}e^{r_{t+1}}\right] \tag{34}$$

$$1 = E_t\left[\beta\overline{R} + \beta\overline{R}(\eta(c_t - c_{t+1}) + r_{t+1})\right] \tag{35}$$

using the steady state equation (19) $1 = \beta\overline{R}$, then,

$$0 \approx E_t\left[\eta(c_t - c_{t+1}) + r_{t+1}\right] \tag{36}$$

4) For equation (17),

$$\log Z_t = (1-\psi)\log\overline{Z} + \psi\log Z_{t-1} + \varepsilon_t \tag{37a}$$

$$\log\left(\overline{Z}e^{z_t}\right) = (1-\psi)\log\overline{Z} + \psi\log\left(\overline{Z}e^{z_{t-1}}\right) + \varepsilon_t \tag{37b}$$

$$z_t = \psi z_{t-1} + \varepsilon_t \tag{37c}$$

To summarize, let us rewrite equations (27), (32), (36) and (37c) as:

$$c_t \approx \frac{\overline{Y}}{\overline{C}}z_t + \frac{\overline{K}}{\overline{C}}\overline{R}k_{t-1} - \frac{\overline{K}}{\overline{C}}k_t \tag{38a}$$

$$r_t \approx (1-\beta(1-\delta))(z_t - (1-\rho)k_{t-1}) \tag{38b}$$

$$0 \approx E_t\left[\eta(c_t - c_{t+1}) + r_{t+1}\right] \tag{38c}$$

$$z_t = \psi z_{t-1} + \varepsilon_t \tag{38d}$$

2.2.3 Solving for the dynamics

Solving for the dynamics by the method of undetermined coefficients is to postulate a linear recursive law of motion between the endogenous variables c_t, k_t, r_t and the state variables k_{t-1}, z_t.

$$k_t = \nu_{kk}k_{t-1} + \nu_{kz}z_t \tag{39a}$$

$$r_t = \nu_{rk}k_{t-1} + \nu_{rz}z_t \tag{39b}$$

$$c_t = \nu_{ck}k_{t-1} + \nu_{cz}z_t \tag{39c}$$

The task is to solve for the coefficients ν_{ij} for $i = \{k, r, c\}$ and $j = \{k, z\}$. Note that these coefficients are the elasticities, i.e., if $\nu_{ck} = 0.5$ and $k_{t-1} = 0.1$ (K_{t-1} is 10% above its steady state value), then $c_t = 0.05$ (C_t is 5% above its steady state value). To solve for these coefficients, one has to substitute the postulated linear law of motion into the left-hand side of equations (38a), (38b), (38c) and (38d). Note that $E_t(z_{t+1}) = \psi z_t$ from applying the linear expectation operator on equation (38d).

1) For the equation (38a),

$$c_t = \frac{\overline{Y}}{\overline{C}} z_t + \frac{\overline{K}}{\overline{C}} \overline{R} k_{t-1} - \frac{\overline{K}}{\overline{C}} k_t \tag{40}$$

$$c_t = \frac{\overline{Y}}{\overline{C}} z_t + \frac{\overline{K}}{\beta \overline{C}} k_{t-1} - \frac{\overline{K}}{\overline{C}} k_t \tag{41}$$

$$\nu_{ck} k_{t-1} + \nu_{cz} z_t = \frac{\overline{Y}}{\overline{C}} z_t + \frac{\overline{K}}{\beta \overline{C}} k_{t-1} - \frac{\overline{K}}{\overline{C}} (\nu_{kk} k_{t-1} + \nu_{kz} z_t) \tag{42}$$

$$\nu_{ck} k_{t-1} + \nu_{cz} z_t = \left(\frac{1}{\beta} - \nu_{kk}\right) \frac{\overline{K}}{\overline{C}} k_{t-1} + \left(\frac{\overline{Y}}{\overline{C}} - \frac{\overline{K}}{\overline{C}} \nu_{kz}\right) z_t \tag{43}$$

Therefore,

$$\nu_{ck} = \left(\frac{1}{\beta} - \nu_{kk}\right) \frac{\overline{K}}{\overline{C}} \tag{44a}$$

$$\nu_{cz} = \frac{\overline{Y}}{\overline{C}} - \frac{\overline{K}}{\overline{C}} \nu_{kz} \tag{44b}$$

2) For equation (38b),

$$r_t = (1 - \beta(1-\delta))(z_t - (1-\rho)k_{t-1}) \tag{45a}$$

$$\nu_{rk} k_{t-1} + \nu_{rz} z_t = (1 - \beta(1-\delta)) z_t - (1 - \beta(1-\delta))(1-\rho) k_{t-1} \tag{45b}$$

Therefore,

$$\nu_{rk} = -(1 - \beta(1-\delta))(1-\rho) \tag{46a}$$

$$\nu_{rz} = 1 - \beta(1-\delta) \tag{46b}$$

3) For equation (38c),

$$0 = E_t \left[\eta(c_t - c_{t+1}) + r_{t+1}\right] \tag{47}$$

$$0 = E_t \left[\eta((\nu_{ck} k_{t-1} + \nu_{cz} z_t) - (\nu_{ck} k_t + \nu_{cz} z_{t+1})) + \nu_{rk} k_t + \nu_{rz} z_{t+1}\right] \tag{48}$$

$$0 = E_t \left[\eta \nu_{ck} k_{t-1} + \eta \nu_{cz} z_t - \eta \nu_{ck} k_t - \eta \nu_{cz} z_{t+1} + \nu_{rk} k_t + \nu_{rz} z_{t+1}\right] \tag{49}$$

$$0 = E_t \left[\eta \nu_{ck} k_{t-1} + \eta \nu_{cz} z_t + (\nu_{rk} - \eta \nu_{ck}) k_t + (\nu_{rz} - \eta \nu_{cz}) z_{t+1}\right] \tag{50}$$

$$0 = \eta \nu_{ck} k_{t-1} + (\nu_{rk} - \eta \nu_{ck}) k_t + \eta \nu_{cz} z_t + (\nu_{rz} - \eta \nu_{cz}) \psi z_t \tag{51}$$

$$0 = \eta \nu_{ck} k_{t-1} + (\nu_{rk} - \eta \nu_{ck}) k_t + ((\nu_{rz} - \eta \nu_{cz}) \psi + \eta \nu_{cz}) z_t \tag{52}$$

$$0 = \eta \nu_{ck} k_{t-1} + (\nu_{rk} - \eta \nu_{ck})(\nu_{kk} k_{t-1} + \nu_{kz} z_t) + ((\nu_{rz} - \eta \nu_{cz}) \psi + \eta \nu_{cz}) z_t \tag{53}$$

$$0 = ((\nu_{rk} - \eta \nu_{ck}) \nu_{kk} + \eta \nu_{ck}) k_{t-1} + ((\nu_{rk} - \eta \nu_{ck}) \nu_{kz} + (\nu_{rz} - \eta \nu_{cz}) \psi + \eta \nu_{cz}) z_t \tag{54}$$

therefore,

$$0 = (\nu_{rk} - \eta \nu_{ck}) \nu_{kk} + \eta \nu_{ck} \tag{55a}$$

$$0 = (\nu_{rk} - \eta \nu_{ck}) \nu_{kz} + (\nu_{rz} - \eta \nu_{cz}) \psi + \eta \nu_{cz} \tag{55b}$$

To summarize, the equations that will solve for the 'undetermined' coefficients are, (44a), (44b), (46a), (46b), (55a) and (55b).

$$\begin{aligned}
\nu_{ck} &= \left(\frac{1}{\beta} - \nu_{kk}\right)\frac{\overline{K}}{\overline{C}} \\
\nu_{cz} &= \frac{\overline{Y}}{\overline{C}} - \frac{\overline{K}}{\overline{C}}\nu_{kz} \\
\nu_{rk} &= -(1-\beta(1-\delta))(1-\rho) \\
\nu_{rz} &= 1-\beta(1-\delta) \\
0 &= (\nu_{rk} - \eta\nu_{ck})\nu_{kk} + \eta\nu_{ck} \\
0 &= (\nu_{rk} - \eta\nu_{ck})\nu_{kz} + (\nu_{rz} - \eta\nu_{cz})\psi + \eta\nu_{cz}
\end{aligned}$$

To solve for the system (6 equations and 6 unknowns), substitute equations (44a) and (46a) into (55a).

$$0 = \left(-(1-\beta(1-\delta))(1-\rho) - \eta\left(\frac{1}{\beta} - \nu_{kk}\right)\frac{\overline{K}}{\overline{C}}\right)\nu_{kk} + \eta\left(\frac{1}{\beta} - \nu_{kk}\right)\frac{\overline{K}}{\overline{C}} \quad (56)$$

$$0 = -(1-\beta(1-\delta))(1-\rho)\nu_{kk} - \eta\left(\frac{1}{\beta} - \nu_{kk}\right)\frac{\overline{K}}{\overline{C}}\nu_{kk} + \eta\left(\frac{1}{\beta} - \nu_{kk}\right)\frac{\overline{K}}{\overline{C}} \quad (57)$$

Then divide the last equation $\eta\overline{K}/\overline{C}$ to get:

$$0 = \frac{-(1-\beta(1-\delta))(1-\rho)\overline{C}}{\eta\overline{K}}\nu_{kk} - \frac{1}{\beta}\nu_{kk} + \nu_{kk}^2 + \frac{1}{\beta} - \nu_{kk} \quad (58)$$

$$0 = \nu_{kk}^2 + \left(\frac{-(1-\beta(1-\delta))(1-\rho)\overline{C}}{\eta\overline{K}} - \frac{1}{\beta} - 1\right)\nu_{kk} + \frac{1}{\beta} \quad (59)$$

$$0 = \nu_{kk}^2 - \left(\frac{(1-\beta(1-\delta))(1-\rho)\overline{C}}{\eta\overline{K}} + \frac{1}{\beta} + 1\right)\nu_{kk} + \frac{1}{\beta} \quad (60)$$

Now rewrite the last equation as,

$$0 = \nu_{kk}^2 - \gamma\nu_{kk} + \frac{1}{\beta} \quad (61)$$

where $\gamma \equiv \left(\frac{(1-\beta(1-\delta))(1-\rho)\overline{C}}{\eta\overline{K}} + \frac{1}{\beta} + 1\right)$. Note that from the steady state relations $\overline{Y} = \overline{Z}\,\overline{K}^{\rho}$ and $\overline{C} = \overline{Y} - \delta\overline{K}$, therefore $\overline{C}/\overline{K} = \overline{Z}\,\overline{K}^{\rho-1} - \delta$. Also from equation (18b),

$$\overline{Z}\,\overline{K}^{\rho-1} = \frac{1/\beta - 1 + \delta}{\rho} = \frac{1-\beta+\beta\delta}{\rho\beta} \quad (62a)$$

$$\frac{\overline{C}}{\overline{K}} = \overline{Z}\,\overline{K}^{\rho-1} - \delta = \frac{1-\beta+\beta\delta}{\rho\beta} - \delta = \frac{1-\beta+\beta\delta-\rho\beta\delta}{\rho\beta} \quad (62b)$$

$$\frac{\overline{C}}{\overline{K}} = \frac{1-\beta+(1-\rho)\beta\delta}{\rho\beta} \quad (62c)$$

Note that γ is defined as,

$$\gamma \equiv \left(\frac{(1-\beta(1-\delta))(1-\rho)\overline{C}}{\eta\overline{K}} + \frac{1}{\beta} + 1 \right) \tag{63}$$

Now replace $\frac{\overline{C}}{\overline{K}}$ into γ to get:

$$\gamma = \left(\frac{(1-\beta(1-\delta))(1-\rho)(1-\beta+(1-\rho)\beta\delta)}{\eta\rho\beta} + \frac{1}{\beta} + 1 \right) \tag{64}$$

Note that $\gamma > 0$. Solving the quadratic equation (61) of ν_{kk} yields,

$$\nu_{kk} = \frac{\gamma \pm \sqrt{\gamma^2 - 4/\beta}}{2} \tag{65}$$

$$\nu_{kk} = \frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2 - 4/\beta}{4}} \tag{66}$$

$$\nu_{kk} = \frac{\gamma}{2} \pm \sqrt{\left(\frac{\gamma}{2}\right)^2 - \frac{1}{\beta}} \tag{67}$$

The product of the two roots is equal to $1/\beta$. The smaller root is the stable one, i.e. smaller than one in absolute value. Therefore,

$$\nu_{kk} = \frac{\gamma}{2} - \sqrt{\left(\frac{\gamma}{2}\right)^2 - \frac{1}{\beta}} \tag{68}$$

Once ν_{kk} is computed, the rest of the coefficients can be derived. Equations (46a) and (46b) compute ν_{rk} and ν_{rz} directly from the parameters. Substitute (68) into (44a) to get ν_{ck}. Plug (44b) into (55b) and solve for ν_{kz}. Finally, replace ν_{kz} by its value in (44b) to get ν_{cz}. For some 'calibrated' quarterly parameters, the coefficients are:

$\beta = 0.990$	$\rho = 0.360$	$\eta = 1.000$	$\delta = 0.025$	$\overline{Z} = +1.000$	
$\nu_{kk} = 0.965$	$\nu_{kz} = 0.075$	$\nu_{ck} = 0.618$	$\nu_{cz} = 0.305$	$\nu_{rk} = -0.022$	$\nu_{rz} = 0.035$

2.2.4 Impulse Response Function (IRF)

The system (39) can be used to graph the impulse response function of the model. First to simulate the model, pick some initial values for k_{-1} and z_0, then generate ε_t. Using the system of $z_t = \psi z_{t-1} + \varepsilon_t$ (equation (37c)) and $k_t = \nu_{kk}k_{t-1} + \nu_{kz}z_t$, generate all the other variables c_t, r_t and y_t. The IRF is traced out by setting $\varepsilon_1 = 1$, and $\varepsilon_t = 0$ for $t > 1$. The effect of such a shock on all variables is then graphed.

2.3 Dynamic Programming

A dynamic programming problem is an optimization problem in which decisions are taken sequentially over a period of time. Usually, decisions taken in any period influence the environment. A 'state' variable represents the environment and moves through time in response to the actions taken by the decision maker. Also, it restricts the actions available to the decision maker at any point of time.

Taylor and Uhlig (1990) compared a set of alternative methods to provide numerical solutions for nonlinear rational-expectations models. The eight solution methods compared were: Value-Function Grid, Quadrature Value-Function Grid, Linear-Quadratic, Backsolving, Extended Path, Euler-Equation Grid, Parameterizing Expectations and Least Squares projections. They showed that different methods do lead to different results. Of the methods mentioned (more on methods later) above, we choose to use the value-function grid. However, a much easier and faster method is the parameterizing expectations method of Den Haan and Marcet (1990). This method uses the first-order conditions for the dynamic problem. A power function approximates the conditional expectation function then a nonlinear regression is estimated on one set of initial parameters. Iteration on the parameters continues until minimization of the mean square error between the power function and the conditional expectation is achieved.

3 Baseline RBC (King, Plosser & Rebelo)

This section underlines and follows the calibration process of the 'Baseline[4]RBC model'[5]as presented in King, Plosser and Rebelo (KPR) (1988a).

The baseline model assumes a single type of output that is consumed or invested. The output is produced by a Cobb-Douglas technology with constant returns to scale. Labour and capital are inputs. Consumers' - infinitely lived agents - preferences are ordered by the time discounted momentary utility over log of consumption and weighted log of leisure. The use of the log is adopted to match the positive trend in real wages and the zero trend in annual hours per worker.

- Preferences

$$E_t \sum_{t=0}^{\infty} \beta^t \left[\ln(C_t) + \theta \ln(L_t)\right] \qquad 0 < \beta < 1 \tag{69}$$

- Technology

$$Q_t = K_t^{1-\alpha} \left[A_t N_t\right]^{\alpha} \qquad 0 < \alpha < 1 \tag{70}$$

and (the impulse)

$$\ln(A_t) \equiv a_t = \gamma_a + \rho_a a_{t-1} + \varepsilon_t \qquad \varepsilon_t \sim iid(0, \sigma_\varepsilon) \tag{71}$$

- Capital law of motion (the propagation)

$$K_t = I_t + (1-\delta) K_{t-1} \qquad 0 < \delta < 1 \tag{72}$$

- Resource constraints

$$C_t + I_t \leq Q_t \tag{73}$$

4 Also, referred to as the 'Benchmark RBC model' in the literature.
5 Reproduced in the appendix with data measurements for the U.S. economy.

and

$$N_t + L_t = 1 \tag{74}$$

To compute the model' predictions, maximize the utility function subject to the technology and the constraints. This equilibrium solution of the model is a function of the parameters which imply a stochastic process for the variables C_t, L_t, N_t, K_t, I_t and Q_t. In general, the solution is a non-linear function of the parameters and there is no closed-form solution. So, numerical methods are needed to calculate the stochastic process for the variables.[6] Approximating the solution by the log-linearization of the Euler equations yields a vector autoregression (VAR) for the logarithms of C_t, N_t, K_t, I_t and Q_t (usually denoted by lowercase letters c_t, n_t, k_t, i_t and q_t). All the variables are non-stationary (except for n_t) and are represented by stationary deviations about a_t, which follows an integrated process by assumption (when ρ_a[7] equation (71), page 9 is equal to one). Therefore, they are cointegrated with a common trend, namely a_t. Note that the coefficients of the VAR are complicated functions of the parameters of the model. Once these values are replaced numerically (calibration), the equilibrium can be generated and the autocovariance generating function of $x_t = (\ \Delta c_t, \Delta i_t, \Delta q_t, n_t')$ follows. Finally, the properties of these artificially generated stochastic processes are compared with the real world data.

The first order conditions for this basic RBC model are

$$U_{lt} = U_{ct} MPL_t \tag{75}$$

i.e., the marginal rate of substitution between leisure and consumption is equal to the real wage rate (under the perfect competition condition). This implies that if the real wage rate increases and the utility function has $U_{ll}, U_{cc} < 0$, then consumption will increase and leisure will decrease. And

$$U_{ct} = E_t \beta (1 + MPK_{t+1} - \delta_k) U_{ct+1} \tag{76}$$

so that consumption growth is related to the net return on capital.

In this setup, the only source of fluctuation in the economy is A_t (which represents technology shocks). A change in A_t will change the quantity of labour demanded. The extent to which employment will be influenced following a shock depends crucially on the labour supply function. More technically, it depends on the intertemporal substitution of labour supply (see section 5 for details). If labour supply is infinitely elastic, then the effect of the shock on employment will be maximized and the real wage will exhibit an acyclical pattern. Note that empirical micro level studies indicate that the labour supply - especially for adult men - is inelastic (vertical) in the long-run.

The choice of parameter values[8] for quarterly variables in the U.S.A. over 1948-1986 is:

- $\alpha = 0.58$, equal to the average value of labour's share of GNP over the period.

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6 Methods of numerical solutions are exhaustively reviewed and compared in Taylor and Uhlig (1990).

7 The first order coefficient of the technology process.

8 King, Plosser and Rebelo (1988b, p. 314 footnote 3).

- $\gamma_a = 1.04$, as the common trend of log per capita values of real GNP, consumption of non-durables and services, and gross fixed investment.[9]

- $\delta = 0.025$, to yield a gross investment share of GNP of approximately 30 percent.

- $\theta = 0.20$, so that the model steady-state value of N (= 0.20) matches the average workweek as a fraction of total hours over the period.

- $\beta = 0.988$, so that the model's steady-state annual interest rate matches the average rate of return on equity over the period. $(\beta \equiv \frac{1}{1+r})$

- $\sigma_\varepsilon = 0.01$, as a convenient normalization.

4 Intuition of RBC

For the baseline model, the predictions following, say, a negative technology shock are of an immediate fall of employment. Output falls because of the direct effect of the decline in employment and the effect of the decline in productivity. Since capital is unaffected, the marginal product of capital and thus the interest rate fall. Consumption - governed by the intertemporal Euler equation - rises and investment declines by more than the decline in output. This is followed by gradual increases in the interest rate and real wage back to their normal levels. As a result, consumption falls back to normal and employment rises back to normal.

After calibrating the King, Plosser and Rebelo (KPR) model, the study concluded that the model successfully reproduced the relative ranking of the variances of consumption, labour hours, investment and output in business cycle data. However, the model failed to reproduce the appropriate stylised fact on the interaction between output and labour hours.

Before presenting varieties of RBC models and their results, I will address two issues pertaining to RBC in general. The first is the importance of elasticities. The second is the issue of de-trending. These issues are important to understand RBC criticisms.

5 Elasticities

Since the extent of the effect of a shock on employment depends on the intertemporal substitution of labour supply, this subsection defines and emphasizes the role of elasticities in the baseline RBC model. For the general momentary utility function,

$$U(C, L) = V(C) \bullet C(L) = \frac{1}{1-\sigma} C^{1-\sigma} \bullet \frac{1}{1-\sigma_1} L^{1-\sigma_1} \tag{77}$$

We have,

$$\begin{array}{ll} U_C = C^{-\sigma} V(L) > 0 & U_L = L^{-\sigma_1} V(C) > 0 \\ U_{CC} = -\sigma C^{-1-\sigma} V(L) < 0 & U_{LL} = -\sigma_1 L^{-1-\sigma_1} V(C) < 0 \\ U_{CL} = U_{LC} = C^{-\sigma} L^{-\sigma_1} > 0 & \end{array} \tag{78}$$

[9] King, Plosser and Rebelo (1988a, p. 226 and footnote 35 on same page).

The elasticities of marginal utility U_C with respect to C and L are,

$$\zeta_{U_c C} = \frac{U_{CC}C}{U_C} = \frac{-\sigma C^{-1-\sigma}V(L)C}{C^{-\sigma}V(L)} = -\sigma \tag{79}$$

$$\zeta_{U_c L} = \frac{U_{CL}L}{U_C} = \frac{C^{-\sigma}L^{-\sigma_1}L}{C^{-\sigma}\frac{1}{1-\sigma_1}L^{1-\sigma_1}} = 1-\sigma_1 \tag{80}$$

The intertemporal elasticity of substitution in consumption equals σ. As σ increases (approaches 1, i.e. logarithmic), the decrease in U_C is more rapid in response to an increase in C, and the consumer is less willing to accept deviations from a uniform pattern of consumption.

The elasticities of marginal utility U_L with respect to C and L are,

$$\zeta_{U_L C} = \frac{U_{LC}C}{U_L} = \frac{C^{-\sigma}L^{-\sigma_1}C}{L^{-\sigma_1}\frac{1}{1-\sigma}C^{1-\sigma}} = 1-\sigma \tag{81}$$

The intertemporal elasticity of substitution in leisure equals σ_1, as shown by,

$$\zeta_{U_L L} = \frac{U_{LL}L}{U_L} = \frac{-\sigma_1 L^{-1-\sigma_1}V(C)L}{L^{-\sigma_1}V(C)} = -\sigma_1 \tag{82}$$

5.1 The Frisch Elasticity of Labour Supply

It is useful to consider the λ-constant or Frisch labour supply. Let λ be the Lagrangian multiplier associated with the worker's intertemporal budget constraint. The first-order condition associated with the labour supply is,

$$\frac{\partial U(n_1, ..., n_t, ...n_T)}{\partial n_t} = \lambda w_t \tag{83}$$

where w_t denotes the real wage in period t stated in period 0 prices (discounted to period 0). The Frisch inverse labour supply function is the marginal disutility of work stated in wage units:

$$\frac{1}{\lambda} \cdot \frac{\partial U(n_1, ..., n_t, ...n_T)}{\partial n_t} \tag{84}$$

When U is additively separable in labour, this can be solved to simplify for the labour supply as a function of the current wage. When U is not additively separable, the supply price of work in one period is a function of the level of work in that and other periods.

The elasticity of the labour supply schedule is

$$\zeta = \sigma_1 \cdot \frac{\overline{n} - n}{n} \tag{85}$$

It is equal to the intertemporal elasticity of substitution in leisure, σ_1, multiplied by the ratio of non-work time to work time. The elasticity ζ controls labour supply over the life cycle. If the wage rate were to double (fully anticipated by the worker at age 20) over the same period, a worker with an ζ of 1 will work twice as many weeks at age 40 as at age 20. Empirical evidence points to ζ being near the values 0.1 to 0.2. A larger Frisch elasticity generates larger responses to economic shocks in equilibrium models, since agents are more willing to substitute leisure across time.

For the utility kernel $(1-\phi)\log C_t + \phi\log(1-n_t)$, the Frisch elasticity of labour supply equals $(1-n)/n$, the steady-state ratio of leisure to labour, or $\phi/(1-\phi)$. The intertemporal elasticity of leisure is equal to 1. A 1% change in leisure results in $\frac{\phi}{1-\phi}\%$ change in hours of employment. This kernel is often criticized that its labour supply elasticity is much higher than that of prime age males estimated from panel data.

Christiano and Eichenbaum (1992) used a range of 3 to 5 for the Frisch elasticity. They estimated ϕ to be equal to 5/6. Prescott (1986) choose a value for ϕ closer to 2/3, but typically magnifies this elasticity by allowing past values of leisure to enter into the utility function. A value of 2/3 means[10] that 2/3 of the time is allocated to non-market activities. Swanson (1999b) used a value of 1.7/3.

Lloyd and Niemi (1979) investigated if the labour supply elasticity shifted over time, and for which demographic groups it did. Using quarterly U.S.A. data,[11] the study found evidence of statistically significant shifts - from the period 1956-1965 to 1966-1976 - in the labour supply elasticity. The most significant shift was due to sectoral shifts in demand, unfavourable to men and favourable to women (i.e., increased female participation rates).

6 De-trending

Lucas' (1977) definition of the term 'business cycle' requires detrending the business cycle data. If 'business cycle' fluctuations are defined as deviations around a trend, then a natural first step to examine the fluctuations is to de-trend the data. One way of eliminating the trend is to use the Hodrick-Prescott filter (Hodrick and Prescott 1980).

The first step of the HP curve-fitting method is to take the logarithms of the variables for two reasons: 1) to compress the units in which the variables are measured in, and 2) because of the inherent exponential trend in most aggregate economic variables. The selected trend path $\{\tau_t\}$ is one which minimizes the sum of squared deviations from a given series $\{Y_t\}$ subject to the constraint that the sum of the squared sum differences not be large. Formally,

$$\min_{\{\tau_t\}_{t=1}^{T}} \sum_{t=1}^{T} (Y_t - \tau_t)^2 \tag{86}$$

subject to

$$\sum_{t=2}^{T-1} \left[(\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1})\right]^2 \leq \mu \tag{87}$$

where μ is a parameter governing the smoothness of the trend. The smaller μ is, the smoother it is. If $\mu = 0$, the least squares time trend is linear. Usually, μ is set so

[10] This is the value we adopt in this thesis.
[11] Source : *Employment and Earnings*.

that the Lagrangian multiplier λ of the constraint equals 1600. When the observation period is in quarterly frequency, this produces the appropriate degree of smoothness. Therefore, the minimization problem reduces to

$$\min_{\{\tau_t\}_{t=1}^{T}} \sum_{t=1}^{T}(Y_t - \tau_t)^2 + \lambda \cdot \sum_{t=2}^{T-1} \left[(\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1})\right]^2 \tag{88}$$

$$\min_{\{\tau_t\}_{t=1}^{T}} \sum_{t=1}^{T}(Y_t - \tau_t)^2 + 1600 \cdot \sum_{t=2}^{T-1} \left[(\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1})\right]^2 \tag{89}$$

The second sum of the squared term is an approximation of the derivative of τ_t at time t. One attempts to minimize two sums of squares: the sum of squared cyclical residuals and the sum of squared $\Delta^2\tau_t$. The smoothing parameter λ gives relative weight to these two sums of squares.[12]This parameter acts as a penalty for the acceleration of growth. Finally, the deviations from trend are computed as,

$$Y_t^d = Y_t - \tau_t \qquad \text{for} \quad t = 1, ...T \tag{90}$$

The HP filter is a high band pass filter that eliminates all frequencies of 32 quarters (8 years) or greater. It decomposes the macroeconomic time series into a nonstationary trend component and a stationary cyclical component. Over the past nineteen years, the HP filter became the standard practice to detrend and the hallmark of real business cycle models.

Proponents of the use of the HP filter often explain that it is just a computational procedure used to fit a smooth curve through the data, i.e., that ‘it is just a curve-fitting technique’. Opponents of the use of the HP filter have shown that the filter distorts the dynamic properties of the data. The filter is responsible for generating spurious business cycle periodicity when there is no cycle present in the original data (see Cogley and Nason (1995)). Also, King and Rebelo (1993) provided examples in which the use of HP filter alters substantially measures of persistence, variability and co-movements of economic time series data. They advocated the implementation of a trend component in RBC models to eliminate the use of any filtering.

There are also other detrending methods in the literature. For example, Lucas (1980b) employed an exponential smoothing filter (ES) in his investigation of the quantity theory of money. The ES filter solves a minimization problem similar to the HP filter. It is

$$\min_{\{\tau_t\}_{t=1}^{T}} \sum_{t=1}^{T}(Y_t - \tau_t)^2 + \lambda \cdot \sum_{t=2}^{T-1} \left[(\tau_t - \tau_{t-1})\right]^2 \tag{91}$$

Note that the parameter λ here penalizes for the changes in the growth component.

[12] The rationale for setting $\lambda = 1600$ is as follows. The parameter $\lambda = \sigma_c^2/\sigma_\tau^2$, where σ_c^2 denotes the variance of the cyclical component and σ_τ^2 denotes the variance of the trend component. Hodrick and Prescott used “... the prior view that a five percent cyclical component is moderately as large as is one-eight of one percent change in the rate of growth in a quarter ...”. Therefore, $\lambda^{1/2} = \frac{5/1}{1/8}$ or $\lambda = 1600$ as a value for the smoothing parameter.

7 Criticisms

For a complete review of RBC controversies, see the series of discussion papers in The Economic Journal (1995). In my view, criticisms of RBC models are classified as ideological, methodological, end-result and goodness-of-fit.

Ideologically, critics attack the built-in Walrasian market clearing foundation as a way of describing markets behaviour, especially that of the labour market. Many economists object to the notion of agents' intertemporal decisions to generate a labour supply. Their argument is as follows. Explaining the Great Depression on the basis that the labour supply is the product of agents' intertemporal decisions, is likely to be unrealistic. To explain the Great Depression using such decisions, the assumption has to be that agents anticipated WWII a decade prior to its start and decided to hold off their supply for labour until the increase for demand generated by WWII. Such a voluntary-unemployment explanation during 1930s is unreasonable. In his criticism, Stiglitz (1986) questioned also capital (i.e., machines) unemployment during the same era.

Methodologically, the criticisms were about the objectivity versus the subjectivity of the calibrating exercise. The use of Solow residuals as impetus came under heavy criticism. The criticisms of the end-result of RBC point to the models' inability to reproduce certain stylised facts such as: variability of employment exceeding that of productivity, the instantaneous correlation between employment and productivity close to zero and average productivity that leads the cycle.[13]Goodness-of-fit criticisms highlighted and strongly condemned the ad-hoc method(s) of judging the merits of each model. The absence of a metric, by which one measures how good is the model as an approximation to the business cycle data, is still a topic of research. Also, the absence of formal statistical tests led many to label the RBC as 'unworthy' of acceptance.

The success of RBC modeling in explaining business cycles is still a question open to debate. However, Eichenbaum (1995, p.1609) reiterated - in defense of the brittleness of RBC - that "We do not need high power econometrics to tell us that models are false. We know that. What we need are interesting diagnostic tools to help us understand the dimensions along which misspecified models do well and the dimensions along which they do poorly".

The real business cycle literature shows that if one is to reconcile the cyclical and persistent pattern of the data with a general equilibrium stochastic macroeconomic model, one must use the same pattern in the 'productivity shocks' that drive the model impulse responses.[14]Empirical measures of aggregate technology are obtained by calculating the Solow (1957) residuals. However, the standard deviation for the U.S.A. Solow residual equals 0.763, while the standard deviation of Gross National Product (GNP) is 1.8 percent. How can one use the 'productivity shocks' (measured by the Solow residual) pattern to drive the model impulse responses and get a result of 1.8 percent variability for GNP?

[13] Usually referred to as 'labour productivity cycle' in the literature.
[14] This is usually referred to as 'The baseline real business cycle model' in the literature.

Cogley and Nason (1993) elaborate on this point. They showed that in a typical (baseline) RBC model,[15]output dynamics are determined by impulse dynamics. In other words, the output series generated from the artificial model is represented as a filtered transformation of the external shocks to the model. For example, if the shock is an AR(1) process, then output is an ARMA(3,2) process. In brief, external shocks completely drive the model's generated output series, pointing out how weak is the propagation mechanism of typical RBC models. The output dynamic properties are only a reflection of the impulse dynamic properties. There are just not enough dynamics (the propagation mechanism is very weak) in the typical RBC models since the output dynamic properties are completely dominated by impulse dynamics.

7.1 Goodness-of-fit

Real Business Cycle research has often relied on matching unconditional second moments from the data in the real economy with unconditional second moments from the data generated by an artificial model economy. Such an approach to assess the model's goodness-of-fit was heavily criticized and labeled - by many - as 'the eye-ball metric'. A classical alternative was suggested by Watson (1993). This study developed a goodness-of-fit measure for the class of dynamic econometric models in which all the endogenous variables are covariance stationary. In this context, the economic model is an abstraction of the real economy and is viewed as an approximation to the stochastic process generating the data. To measure the quality of this approximation, Watson proposed a measure of goodness-of-fit motivated by models of measurement errors in the Slutsky (1927) spirit. His approach was to quantify how much stochastic error must be added to the model's variables so that the model's artificial second moments do match the real economy' moments. This treats the discrepancy between the model and data as a stochastic process. Once this error is computed, one can construct a measure of fit from its size. This approach to minimizing the approximation error, in a sense, mirrors the R^2 in simple linear regression.

The criticisms of Watson' procedure are: 1) it can not account for moments other than the second ones and 2) nonlinearities and variations in conditional second moments (such as ARCH type time series) are ignored for simplicity. Another criticism is on how the procedure views the parameters. In the usual calibration exercises parameter values are viewed as a point-mass priors around the values.

A different procedure was proposed by Bayesian analysis. DeJong et al (1996) proposed a Bayesian approach to deal with the parametric uncertainty. By specifying a prior distribution over the parameter values, one can generate a distribution over the statistical properties of the simulated artificial data. In the case of the typical RBC model, this procedure concluded that modest prior specification is the road to take.

In general, the ratio of the standard deviations of aggregate hours to those of output has been emphasized in the literature as a measure of the simulated model economy's

[15] The Cogley and Nason (1993) typical RBC model is in the appendix, as well as the parameters values used in their study.

goodness of fit (see Kydland and Prescott 1982 and Hansen 1985). It is 1.47 for the U.S.A. data.

8 Varieties of RBC

This section presents different varieties of RBC models and their results discussed in the literature.

8.1 Indivisible Labour

This approach,[16]developed by Hansen (1985), assumes that all variations in employment happen at the extensive margin. It creates a highly elastic labour supply at the aggregate level irrespective of the elasticity of the individual agents. Formally,

$$n_t = h_0 \phi_t \tag{92}$$

where ϕ_t is the proportion of individuals working and h_0 is the fixed shift length. In this setup, the agent works or does not work because of fixed costs.[17]After using a lottery to determine which individuals are working and make the preference space[18]convex, the model draws on a linear utility function in employment. This setup supports a constant marginal utility of leisure regardless of the hours worked and gives rise to a highly elastic labour supply. Individuals can either work or not work. As a result, this framework does not add persistence in unemployment, but accounts for employment volatility.

8.2 Labour hoarding

The focus in this model is on the intensive margin. Developed by Burnside et al (1993), this model kept the Walrasian essence and added a sequential decision-making tree. The production function used is

$$y_t = \theta_t k_{t-1}^{\alpha} \left(e_t h_0 n_t\right)^{1-\alpha} \tag{93}$$

where e_t, h_0 represent the effort and the work shift length respectively. The product $e_t.h_0$ represents the labour supplied by the individual. Labour hoarding increases as e_t goes to zero, and diminishes to zero when operating at full capacity ($e_t = 1$). In this model, leisure is specified by $T - \chi - e_t.h_0$ where T and χ are the time endowment and the fixed cost of hiring respectively.

The sequential decision process is as follows. The firm chooses N_t prior to the realization of the shock θ_t, then chooses e_t after the shock. The firms' decisions on hiring and firing are based on the expected productivity shock and it can adjust the amount of labour demanded only in the following period. By persuading workers to

[16] Such methodology have proven successful results when confronted with U.S. data but failed when European data was in question. This approach is to endogenize the labour supply.

[17] For example commuting time.

[18] Since the preference space is binary, to work or not to work.

contribute more effort following a positive shock, the firm focuses on the intensive margin and effort is adjusted to clear the labour market.

Boileau and Normandin (1997) concluded that the labour hoarding model provided a better account of employment dynamics than alternative models applied to the U.S.A. data.

8.3 Search

In this framework, Pissarides (1990) and Merz (1995) produce persistence using a non-Walrasian analysis in the sense that the marginal productivity of labour is not set equal to the real wage. The key in these models is a matching function of unemployed workers to firms added to the law of motion of employment, which is

$$n_{t+1} = (1 - \delta_n)n_t + m_t \tag{94}$$

where δ_n is the proportion of the outflow from employment to unemployment and m_t is the matching function of the new hires. In these models, the Beveridge curve represents the matching function between unemployment and vacancies. The matching function is:

$$m_t = Av_t^{\eta}(1 - n_t)^{1-\eta} \tag{95}$$

where A is the efficiency of labour market clearing, v_t is the number of vacancies and $1 - n_t$ is the number of unemployed. The distribution of income created from the matching function depends on the firm's monopoly power and the workers' ability to bargain. The supply of vacancies determined by the firms depends positively on the level of unemployment. The higher is the unemployment, the easier it is to fill jobs at a lower cost.

8.4 Results of all Varieties of RBC models

In an attempt to explain unemployment persistence in the U.K., Millard,[19] Scott and Sensier (1999) simulated all of the above varieties of RBC models. The business cycle data are quarterly for the U.K. covering the period from 1976:Q2 to 1996:Q2. Their simulated results for the standard deviation of the following variables relative to output standard deviation and compared to the business cycle data are:

	Consumption	Investment	Employment	Unemployment
Business Cycle Data	0.97	2.47	1.11	8.43
Basic RBC	0.38	1.42	0.22	0.12
Indivisible Labour	0.83	3.05	0.36	0.20
Labour Hoarding	0.32	1.61	0.42	0.25
Search	0.87	1.48	0.22	0.13

Source: Millard, Scott and Sensier (1999, p. 26).

Across all models, the basic RBC performs the worst in terms of replicating employment and unemployment variability. Labour hoarding provides the best performance

[19] I would like to thank Stephane Millard for providing the computer codes.

for both variability. However, the suggested values are much lower than the respective business cycle data. The conclusion is that all models generate low volatility in either employment or unemployment and cannot explain the observed persistence of U.K. unemployment.[20]

8.5 Labour Adjustment Costs

Riddell (1999, p. 24) acknowledged employer adjustment costs as an explanation for the high unemployment in Europe. Amano and Macklem (1998) estimated a dynamic linear quadratic model of aggregate labour demand for Canada, the U.S.A. and Germany. They concluded that the adjustment costs of the labour demand are very similar in Canada and the U.S.A. and are an important source of employment fluctuations.

In general, adjustment costs occur when it is costly for firms to adjust employment. Employment persists for many periods and sluggishly adjusts following an economy wide technology shock. Given the inability of the above varieties of RBC in simulating unemployment persistence or adequate employment volatility, and the Riddell (1999) acknowledgment, we will investigate two derivatives of RBC models (impulse mechanism) that include labour adjustment costs. Note that by its nature, the adjustment costs will induce smaller employment volatility. When faced with a cost (in terms of lost leisure) to reallocate, representative agents will not to change employment across sectors so frequently as without costs. In theory, adjustment costs (i.e., propagation mechanism) are a useful means of generating unemployment persistence.

9 RBC failures

This section focuses on RBC failures to account for observed employment variability and output persistence.

9.1 Observed employment volatility

Prescott (1986) reported that observed employment is twice as volatile as the one simulated from the standard RBC economy. In the U.S.A. data, the variance of hours worked relative to the variance of output equals 0.95 percent. A usual RBC baseline model generates a ratio of 0.52 percent. Most RBC models generate a substantially smaller volatility in employment than that in the data.

Campbell (1994) investigated this issue. The study found that a one percent shock, decreasing technology, lowered employment by 0.45 percent in the baseline RBC model. Therefore, to explain a decline of three percent employment in recession, one must assume a seven percent decrease in technology, a number which is obviously unrealistic. For Europe, employment did not rise during the 1970-1985 period, although total factor productivity increased more than twice as much as it did in the U.S.A. Failure of RBC models to generate matching employment variability sparked wide interest among

[20] Results for persistence are not replicated here. Please refer to Millard, Scott and Sensier (1999, p. 27-32) for the full analysis.

researchers and led to a search for alternatives that could explain this observation. Examples included: indivisible labour, nominal wage contracts and labour market search.

Fraisie and Langot (1994, p. 1581) asked the same question: "Can RBC models be saved?" They considered a model with indivisible labour, labour hoarding and adjustment costs. They concluded that "... the introduction of labor adjustment costs is a necessary condition for the model to reproduce a productivity cycle ..." (p. 1582). However, they also concluded that labour hoarding is a necessary assumption to achieve a one period gap between productivity and employment (in business cycle data, employment is coincident and productivity is leading).

9.2 Persistence in aggregate output

Cogley and Nason (1995) concluded that actual output dynamics are more persistent than those generated from standard RBC models. Since the baseline KPR model is driven only by the single technology shock, the persistence of the output, consumption and investment depended heavily on the persistence assumption used in the technology shock.

The baseline model fails to account for the heterogeneity of the workers or jobs. It does not contain incentives for a worker to change jobs and no suggestion that a worker might be more productive in the new job than the current one. The focus here is on the movement of workers from unsuccessful productive units to growing ones.

10 Adjustment Costs and the autocorrelation of output

One dimension in which adjustment costs are useful is in matching of the autocorrelation function of output growth. A weakness (among many, see section 7) of the baseline RBC is its inability to predict (match) the positive serial autocorrelation in business cycle output growth rates. In the U.S.A. data, real output growth rates are positively serially correlated and the serial autocorrelation is significantly higher than zero for lags of one and two quarters (see Cogley and Nason 1995). This discrepancy between model generated and business cycle data is present in a wide class of RBC models.

	1st autocorrelation	**2nd autocorrelation**
Data	0.37	0.22
King et al model (1988b) (KPR)	0.02	0.02
Schmitt-Grohé model (1998)	0.18	0.12

Using a two-sector RBC model, and a random walk technology shock, Grohé-Schmidt (1998) focused on the matching of the autocorrelation function of output growth rates. The investment sector and the consumption goods sector were characterized by increasing returns and constant returns to scale, respectively.

Most RBC models correctly predicted this positive autocorrelation. These different models used a wide variety of assumptions: e.g., employment lags in the labour hoarding process, such as in Burnside, Eichenbaum and Rebelo (1993), adjustment costs in factor inputs, an AR(2) technology shock or government shock.

11 Aggregate Returns to Scale and RBC

Cole and Ohanian (1999) questioned the sensitivity of RBC models to the parametric form and the value of the aggregate returns to scale.[21]

If aggregate returns are constant or decreasing, then there is no mechanism by which a monetary-shock driven extension model of RBC can reproduce the procyclical labour productivity stylised fact. In the simple case where the monetary transmission mechanism holds in the model, a monetary shock will induce an increase in employment (movement along the marginal productivity of labour) without shifting the demand for labour. Therefore, labour productivity will not be procyclical. Note that when the production function exhibits constant or decreasing returns to scale, a technology shock (a supply shock) generates procyclical labour productivity.

However, if aggregate returns are increasing, a monetary shock driven RBC can generate the procyclicality of labour productivity. In the case where the per capita production function of the household depends on the aggregate per capita output (as externality),[22]then a monetary shock generates procyclical labour productivity. Note that if the value of the increasing returns is large, then the model equilibria might not be unique. See for example Benhabib and Farmer (1994) where Keynesian type 'animal spirits' generate business cycle fluctuations.

Since a monetary shock is not the focus, the models proposed do not attempt to include a monetary or a fiscal sector. I will adopt a constant returns to scale production function.

12 Calibration and the Canadian Economy

Calibration originated in the computable general equilibrium (CGE) modelling. Early calibration methods required setting an equilibrium point (as a benchmark) in the product space of the model variables and linearizing a non-linear system around it. In the general equilibrium (GE) setting, calibration became the quantification of unknown parameters either by using micro-level data estimates (plug-in estimates) or by just fixing the parameters (backward reasoning) such that the model produces a steady state within a given interval. For these quasi-scientific practices, calibration has long been a subject of debate in the economic profession.[23]In brief, calibration is the process of choosing parameter values based on microeconomics evidence.

To calibrate the models proposed, I refer to the pioneering work of Goldstein (1998). His study examined the projections of Canadian long-term economic growth prepared by various forecasters. By reporting values for the basic components that make up a potential output projection, Goldstein discussed how the assumptions made (by differ-

[21] Aggregate returns to scale are defined as the percentage of the change in output relative to the percentage change in factor inputs.
[22] In the model, the value for the externality parameter determines aggregate returns to scale.
[23] For an excellent exposition of the merits of calibration versus estimation, see Quah (1995), and for the statistical aspects of calibration in macroeconomics, see Gregory and Smith (1993).

ent forecasters' institutions) in the estimation process, impacted on the projection as a whole. Based on formal and well documented models, Goldstein (1998) reported the estimates of the following institutions: The Conference Board of Canada (CBoC), the University of Toronto's Fiscal and Economic Analysis Program (PEAP), DRI-McGraw Hill (DRMG), Informetrica (Info) and the Department of Finance (DoF).

12.0.1 Growth Accounting

In general, potential output is estimated using principles of growth accounting. The ratio of actual output relative to potential output is useful for fiscal and monetary policies. It gives an indication of demand pressure on the economy. Once the trend in real gross domestic product (GDP) is identified, one can project potential output. There are two approaches to identifying the historical trend in real GDP.

The time series approach to find the trend in GDP data involves a simple regression of real GDP on a time trend (linearly or non-linearly) or using the Hodrick-Prescott filter. One criticism of such an approach is that it is not possible to identify historically the sources of the trend movements (example: the impact of the aging or the growth rate of the population over the trend growth rate) and therefore difficult to forecast. Another criticism regarding the HP filter concerns the reliability of the end points.

An alternative approach is to use a macro model. Using a formal production function, one can break the level of output into different components. The levels of these components are then de-trended and forecasted over time. The projected trend levels are then introduced into the production function to form a potential real GDP projection. The former approach is referred to - in the literature - as a 'Top-Down' approach, while the latter is referred to as a 'Bottom-Up' approach.[24]

The Cobb-Douglas function with a constant returns to scale assumption to model output is,

$$Y_t = A_t.K_t^{\alpha}.L_t^{1-\alpha} \tag{96}$$

where Y, K, L denote output, capital stock and labour input respectively. A is the total factor productivity (TFP) and reflects the Hicks-neutral technological change. Usually, A is estimated as the residual[25](the amount of output not accounted for by either capital or labour). Under the assumptions of perfect competition and constant returns to scale production function, α denotes the capital share in income (nominal GDP at factor cost). α is the elasticity of output with respect to capital. Taking logs of both sides of equation (96) and differentiating yields the growth rates equation (lower case letters represent the log)

$$\frac{\Delta a}{a} = \frac{\Delta y}{y} - \alpha\frac{\Delta k}{k} - (1-\alpha)\frac{\Delta l}{l} \tag{97}$$

where $\frac{\Delta a}{a}$ denotes the growth rate of the TFP and is the estimated residual using actual real GDP and the actual values of the inputs. Note that this estimate of the residual may differ from the true one if one uses a misspecified assumption on the returns to scale

24 Goldstein (1998, p. 144).

25 A is referred to as Solow' residuals since Solow (1957).

of the economy. Also, equation (97) is useful in computing labour productivity (the growth in output per worker) as follows,

$$\frac{\Delta y}{y}-\frac{\Delta l}{l}=\frac{\Delta a}{a}+\alpha\left(\frac{\Delta k}{k}-\frac{\Delta l}{l}\right) \tag{98}$$

For the Canadian economy, various estimated shares in income values over 1980-1996 are:[26]

	Finance	**CBoC**	**PEAP**	**DRMG**	**Info**
Labour	0.64	0.61	0.70	0.62	0.605
Capital	0.31	0.39	0.30	0.32	0.395

Source: Goldstein (1998, p.149).

From the table, note that the weights do not sum to one in two cases: Finance and DRMG. Because both assume a third factor of production, namely natural resources for Finance and energy consumption for DRMG. Also, CBoC and PEAP use the units of workers in measuring labour input in contrast to the units of hours worked used in the Finance and DRMG studies. The above table is very useful in calibrating the income shares parameters for the Canadian aggregate production function.

The labour input is measured either in terms of hours worked or workers. It is made of: a) the labour force[27] source population (the most important changes in which are due to the changes in fertility rates[28] and immigration rates[29]), b) the aggregate participation rate (changes due to the aging of the population), c) the assumed natural unemployment rate (Finance, CBoC and DRMG estimates are 8.9, 7.4 and 8.0 respectively.) and d) a measure of average hours worked per worker. Note that the decomposition of labour input measure depends on the unit in which it is measured.

The capital input is usually decomposed into machinery/equipment and non-residential construction used in non-government commercial activity. DRMG includes the federal and provincial/local governments' capital stock as well. The capital input is computed using a standard accumulation rule $K_t = I_t + (1-\delta)K_{t-1}$. There are different measures for the capital stock depending on the depreciation assumptions used and depending on the level of disaggregation used.[30] Two approaches are proposed to measure aggregate capital: the linear aggregator and the Cobb-Douglas aggregator. For the former, data on aggregate capital is generated by summing up machinery/equipment (me) and non-residential construction (nr) capital. This implies that these both types of capital are perfect substitutes and therefore have infinite elasticity of substitution (DRMG, CBoC and Info). However for the latter, using a Cobb-Douglas functional specification ($K_{total} = K_{m/e}^{0.4} K_{nr}^{0.6}$),[31] these types of capital have elasticity of substitution equal to

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[26] The Conference Board of Canada (CBoC), the University of Toronto's Fiscal and Economic Analysis Program (PEAP), DRI-McGraw Hill (DRMG), Informetrica (Info) and the Department of Finance (DoF).

[27] The labour force is computed by multiplying the source population by the aggregate participation rate.

[28] The fertility rate was around 1.7 so that, on average, a woman living to the age of 45 had 1.7 children.

[29] In Canada, immigration averaged a 250,000 per year during 1997.

[30] CBoC uses more disaggregated data than Finance and PEAP. Info divides capital stocks into 75 industries.

[31] The weights used to compute the share of total capital income are determined as follows. The share

one.

In Canada, the depreciation rates broken by category over the period 1980-1996 are:

	Finance	**CBoC**	**PEAP**	**DRMG**	**Info**
Total	5.9	6.1	8.9	5.5	5.3
Mach/Equip	12.2	35.6	15.5	12.3	6.8
Non-Res	3.2	1.9	4.9	3.5	4.7

Source: Goldstein (1998, p.169).

12.1 Canadian preferences for work

This section is on calibrating the Canadian leisure weight in the utility function.

Drolet and Morissette (1997) investigated the Canadian Survey of Work Arrangements 1995 data to test if work redistribution would eliminate unemployment. Their study (Table 2 , page 19) shows preferences to work (fewer, same or more hours) of employees by industries . This is taken into account when I calibrate the intertemporal substitution of labour parameter in the RBC models. Preferences for work time in percentage as well as the average hours spent on the job by industry are reproduced in the next table.

	MEN				**WOMEN**			
Industry	fewer	same	more	hours	fewer	same	more	hours
Agriculture	2.8	72.0	25.3	48.5	—	—	—	—
Forestry and mining	4.9	75.9	19.2	44.6	—	—	—	—
Construction	2.4	63.3	34.3	41.8	—	—	—	—
Agriculture, et above.	—	—	—	—	9.2	77.1	13.7	35.9
Manufacturing	5.2	71.9	23.0	41.1	8.4	69.1	22.5	38.1
Distributive Services	5.9	66.9	27.2	41.7	8.1	69.6	22.3	35.7
Business Services	5.0	67.6	27.4	40.5	8.6	69.7	21.7	35.5
Consumer Services	3.7	57.7	38.7	39.0	3.4	56.2	40.4	32.0
Public Services	7.9	70.7	21.4	39.4	9.3	67.0	23.6	33.5

Source: Drolet and Morissette (1997).

They concluded that most Canadians would prefer to work longer rather than shorter hours. Those who would prefer shorter hours are professionals at the higher quartile of earnings. Based on their conclusion, I calibrate the leisure weight parameter for the representative agent for the Canadian economy.

13 General Equilibrium (GE) Framework

GE models are not able to account for the persistence in aggregate level output and

of machinery/equipment equals the sample average of $UC_{m/e}K_{m/e}/(UC_{m/e}K_{m/e} + UC_{nr}K_{nr})$, where $UC_{m/e}$ denotes the user cost of machinery/equipment. This user cost is a function of the price deflator, the tax credit, the depreciation rate, the expected inflation, the corporate tax credit and the real interest rate. This function implies that if the user cost of an input falls, it will lower the marginal product of the input.

unemployment. My interest is at the sectoral disaggregated industries level. The plan is to use general equilibrium artificial economies associated with several labour market institutions to account for aggregate employment behaviour.

Within a multi-sector framework, Dupor (1996) considered the aggregate effects of sector-specific shocks to production. This study concluded that the law of large numbers - implying that positive shocks in some sectors are offset by negative shocks in others - applies and that such a modelling strategy is unnecessary to explain the business cycle character. More generally, if there are many independent shocks and labour were mobile between sectors, then the law of large numbers implies that their effect on the aggregate economy would average out to zero. The method used was to introduce interaction between sectors by an input-use matrix in a general equilibrium framework. Dupor's model assumed that every sector sells some intermediate inputs to some other sectors in the economy and that all sectors are *equally important*.

Several recent studies in the literature suggested different mechanisms by which the law of large numbers can be weakened. Mechanisms such as asymmetries, threshold effects, non-linear settings and monopolistic competition have proved useful (see Boldrin et al (1990) and Scheinkman (1990)) in modelling the effects of inter-sectoral shocks on the aggregate level.

For example, building on the Dupor model, Horvath (1997) simulated greater aggregate volatility from sector-specific shocks. He avoided the law of large numbers and assumed that *some sectors are more important* input-suppliers than others. This assumption relies on the relative sizes of the sectors.

14 Appendix: U.S.A. Business Cycle Data

This appendix reports descriptive moments for the U.S.A. business cycle data and their definitions, in the literature.

14.1 Bils and Cho (1994)

Bils and Cho (1994) reported the following summary of U.S.A. data statistics. The data are taken from Citibase, quarterly covering the period from 1955:3 to 1984:1 All series are logged, detrended using the Hodrick-Prescott filter.

Series	StDev	Correlation with Output
Y	1.74	1.00
C	1.29	0.85
C1	0.81	0.65
I	8.45	0.91
K	0.63	0.05
Q	1.74	0.77
H	0.46	0.76
N	1.50	0.81
Y/H	1.18	0.35

Source: Bils and Cho (1994).

Where Y denotes real GNP, C denotes consumption of nondurable and services, C1 denotes the consumption series used by Christiano, which equals C plus the flow of services from durable goods. I is for gross private domestic investment and K is the nonresidential equipment and structures. Q is aggregate hours measured as hours of all persons. H is weekly hours per person at work. N is for all persons at work, i.e., total employment. Finally, Y/H denotes labour productivity measured as output divided by aggregate hours.

14.2 Prescott (1986)

Prescott (1986) reported the following descriptive statistics for the cyclical behaviour of the U.S.A. economy. All series are measured as deviations from trend covering the period from 1954:Q1 to1982:Q4.

Variable	StDev	Cross Correlation with GNP		
		x_{t-1}	x_t	x_{t+1}
GNP	1.8%	0.82	1.00	0.82
Personal Consumption				
Services	0.6	0.66	0.72	0.61
Nondurables goods	1.2	0.71	0.76	0.59
Fixed Investment	5.3	0.78	0.89	0.78
Nonresidential	5.2	0.54	0.79	0.86
Structures	4.6	0.42	0.62	0.70
Equipment	6.0	0.56	0.82	0.87
Capital Stocks				
Total Nonfarm Inventories	1.7	0.15	0.48	0.68
Nonresidential Structures	0.4	-0.20	-0.03	0.16
Nonresidential equipment	1.0	0.03	0.23	0.41
Labour Input				
Nonfarm Hours	1.7	0.57	0.85	0.89
Average Weekly Hours in manufacturing	1.0	0.76	0.85	0.61
Productivity (GNP/Hours)	1.0	0.51	0.34	-0.04

Source: Prescott (1986)